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The Domain Dependence of Dominant Fredholm Eigenvalues for the Helmholtz Operator

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In this article we study the behaviour of dominant Fredholm eigenvalues for the Helmholtz operator in a regular bounded open set Ω in R^m relative to some larger set Ω' if the latter is altered. It is shown that if the frequency is suitably chosen, then the dominant Fredholm eigenvalues decrease when Ω' is decreased. This property was so far merely established for the Fredholm eigenvalues for the Laplacian (Kress and Roach, *J. Math. Anal. Appl.* **55** (1976), 102–111). The results obtained will be applied to improve the convergence of a Neumann–Liouville bounded integral operator series, which serves as a tool in determining the solution of the Dirichlet problem.

1. INTRODUCTION

Let Ω be a bounded open set in R^m $(m \geqslant 2)$ whose components have pairwise disjoint closures. We assume the boundary Γ of Ω to be an $(m-1)$-dimensional C^2-manifold which is positively orientated by the normal field n pointing out of Ω on each of its boundary components. Since Γ is differentiable, there are only finitely many components of Ω and each component is finitely connected. Let

$$X^2(\Omega) := \left\{ u \mid u \in C(\Omega), \int_\Omega |u|^2 < \infty \right\},$$

where $C(\Omega)$ denotes the linear space of continuous functions (for the present without norm), and where the integral is understood in the Riemannian sense and may be improper. Let $(\ ,\ )$ and $\|\cdot\|$ denote the usual scalar product and norm in $X^2(\Omega)$, respectively. The negative Laplacian in $X^2(\Omega)$ is defined by

$$A := (-\Delta) \restriction F(\Omega),$$

where $F(\Omega)$ is a dense subspace of $X^2(\Omega)$ to be specified later. We write A_Ω

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instead of A if we want to emphasize that A depends on Ω. With the operator A we associate the Neumann–Poincaré operator L on $X^2(\Gamma)$ defined by

$$Lf(y) := \int_\Gamma \frac{\partial}{\partial n} \gamma(\cdot, y) \cdot f \qquad (f \in X^2(\Gamma), y \in \Gamma), \tag{1}$$

where γ is the singularity function for A (cf. [3, p. 30]). If $\mu \in \sigma_p(L)$ (the point spectrum of L), then μ is commonly called a "Fredholm eigenvalue for A." By a "dominant Fredholm eigenvalue" we understand a Fredholm eigenvalue of largest modulus.

In a classical paper of Weyl and Courant (cf. [2, p. 1527]), it was shown that for a simply connected plane domain Ω the Fredholm eigenvalues monotonously increase if Ω is enlarged. This is in contrast to the behaviour of the eigenvalues of $A \restriction C_0^\infty(\Omega)$, which monotonously decrease if Ω is enlarged. The domain dependence of Fredholm eigenvalues was used by Warschawski [10] to assess the rate of convergence of the Neumann–Liouville series representing the resolvent of L at the point $+1$ (corresponding to the interior Dirichlet problem). In order to adjust the convergence of a series of iterated boundary integral operators involved with the Dirichlet problem, Kress and Roach [5] proved a similar monotony property of the dominant Fredholm eigenvalues for A relative to some regular bounded simply connected domain Ω' in R^m $(m \geqslant 3)$, where Ω' contains $\bar{\Omega}$ (written, $\Omega' \Supset \Omega$). These authors replace the singularity function γ in (1) by the Green function of $A_{\Omega'}$ (pertaining to Ω'). The eigenvalues of the new operator L will henceforth be termed "Fredholm eigenvalues for A relative to Ω'." Then, if $\Omega' \Supset \Omega'' \Supset \Omega$, and if μ' and μ'' are Fredholm eigenvalues for A relative to Ω' and Ω'' respectively, Kress and Roach demonstrated that $\mu' > \mu''$.

Let us now consider the Helmholtz operator

$$A := \tau \restriction F(\Omega) \qquad (\tau := -\Delta - \lambda, \lambda \in R). \tag{2}$$

If $\lambda > 0$, one can not expect a result as was established by Weyl and Courant, because the Fredholm eigenvalues for the Helmholtz operator are generally complex not real numbers. For sufficiently small $\lambda > 0$, however, certain dominant Fredholm eigenvalues relative to $\Omega' \Supset \Omega$, which are still to be defined, are real. In the present paper we shall demonstrate that they behave like those for the Laplacian.

2. Fredholm Eigenvalues for the Helmholtz Operator

In the following we briefly explain several notations which are required for our work. Let

$$F(\Omega) := \{u \mid u \in C(\bar{\Omega}) \cap C^1(\Omega), \Delta u \in X^2(\Omega)\},$$

$$G(\Omega) := \{u \mid u \in F(\Omega), (\partial/\partial n)\, u \in C(\Gamma)\},$$

and let $G_0(\Omega)$ be the subset of functions in $G(\Omega)$ vanishing on Γ. We require one further function space, which we define by

$$M(\Omega) := \{a \mid a\colon \Omega \to R^m, a \in C(\bar{\Omega}) \cap C^1(\Omega), \nabla \cdot a \in X^2(\Omega)\}.$$

These linear spaces will now be employed to set up certain operators and forms. We have already introduced the Helmholtz operator A by (2) as a map in $X^2(\Omega)$ with dense domain $F(\Omega)$. Furthermore, we set

$$S := (-\Delta) \restriction G_0(\Omega),$$

and define, for given $\lambda \in R$, a Riccati operator $T\colon M(\Omega) \to X^2(\Omega)$ by

$$Ta := \nabla \cdot a - a^2 - \lambda \qquad (a \in M(\Omega)).$$

Preparatory to specifying certain integral operators associated with A, we need to point out some properties of the operator T and the formal differential operator τ defining A. It was shown in [1] that there is a positive function $w \in C^2(\bar{\Omega})$ satisfying $T(-(1/w)\,\nabla w) > 0$ on $\bar{\Omega}$ if and only if $\lambda < \lambda_1(S)$, where $\lambda_1(S) > 0$ denotes the first eigenvalue of S. We take w as a fundamental mode relative to some open set $\Omega^* \Supset \Omega$. Then we define in Ω a formal differential operator

$$\rho_w := w(\nabla/i)(1/w)$$

as well as its formal adjoint

$$\rho_w^* := \rho_{1/w},$$

and further introduce a modified normal derivative by

$$\partial u(x) := \lim_{\eta \to 0_+} i\rho_w u(x - \eta n(x)) \cdot n(x) \qquad (u \in G(\Omega), x \in \Gamma). \tag{3}$$

Of particular interest for our work is the Jacobi factorization

$$\tau = \rho_w^* \rho_w + h^{1/2} h^{1/2},$$

where

$$h := T(-(1/w)\,\nabla w).$$

This factorization is closely related to the modified first Green formula

$$\int_{\Omega} Au \cdot \bar{v} = D(u,v) + \int_{\Gamma} (-\partial u) \cdot \bar{v} \qquad (u, v \in G(\Omega)), \tag{4}$$

where

$$D(u,v) := \int_{\Omega} \rho_w u \cdot \overline{\rho_w v} + \int_{\Omega} h^{1/2} u \cdot h^{1/2} \bar{v} \qquad (u, v \in G(\Omega)) \tag{5}$$

is a modified Dirichlet integral. Henceforth, we abbreviate

$$D(u) := D(u,u) \qquad (u \in G(\Omega)).$$

Clearly, the latter is only meaningful if $\lambda < \lambda_1(S)$, in which case it defines a positive definite quadratic form on $G(\Omega)$.

Let us now choose an open set $\Omega' \Supset \Omega$ (with C^2-boundary) within an ε-neighbourhood of Ω in such a way that

$$\lambda \notin \sigma(S_{\Omega'}), \tag{6}$$

where $\sigma(S_{\Omega'})$ denotes the spectrum of $S_{\Omega'}$. We further assume the boundary of Ω' to be positively orientated by the exterior normal field, and define a set X' as the complement of $\bar{\Omega}$ relative to Ω'. Condition (6) above has been stipulated in order to ensure that the operator $A_{\Omega'}$ possesses a Green function γ', which is obviously uniquely defined. We denote by $\partial\gamma'(\cdot, y)$ and $\partial_y \gamma'(\cdot, y)$ its modified normal derivatives on Γ according to (3), when $y \in \Gamma$ and γ' is finite. Here, ∂ is composed with w appropriate to Ω'.

Finally, we are in a position to set up, for $\lambda < \lambda_1(S_{\Omega'})$, the integral operators associated with A_{Ω} relative to Ω' as maps K and L on $C(\Gamma)$ respectively by

$$Kf(y) := \int_{\Gamma} \partial_y \gamma'(\cdot, y) \cdot f, \tag{7}$$

$$Lf(y) := \int_{\Gamma} \partial\gamma'(\cdot, y) \cdot f \qquad (f \in C(\Gamma), y \in \Gamma). \tag{8}$$

Additionally, we introduce a single-layer potential by

$$Vf(y) := \int_{\Gamma} \gamma'(\cdot, y) f \qquad (y \in \Omega'),$$

and furthermore set

$$Jf(y) := Vf(y) \qquad (y \in \Gamma). \tag{9}$$

The eigenvalues of L will, similarly to the potential–theoretic case, be called "Fredholm eigenvalues for A relative to Ω'." Note that term (7) is decomposable according to

$$Kf(y) = \int_\Gamma \left\{ \frac{\partial}{\partial n_y} \gamma'(\cdot, y) - \frac{1}{w(y)} \left(\frac{\partial}{\partial n} w(y) \right) \gamma'(\cdot, y) \right\} f \qquad (f \in C(\Gamma), y \in \Gamma),$$

and term (8) can be decomposed similarly. This shows that both operators K and L have weakly singular kernels, and are therefore compact in $C(\Gamma)$ endowed with the maximum norm (cf. [7, p. 21]). Since the operators are also mutually adjoint in $X^2(\Gamma)$, we conclude by a result of Jörgens [4, Theorem 5.18] that

$$\sigma_P(L) = \sigma_P(K). \tag{10}$$

On the basis of (10) one can prove the desired monotony property of dominant Fredholm eigenvalues by verifying it for the dominant eigenvalues of K. In the course of this proof we will essentially make use of the fact that term (9) defines a positive definite bounded operator in $X^2(\Gamma)$ as was demonstrated in [1].

3. Monotone Domain Dependence of Dominant Fredholm Eigenvalues

We shall achieve our goal essentially with the aid of two methods. First, we use an extended version of one part of Dirichlet's principle, and second, as an integral component, we use the fact that the operator K permits a symmetrization for all λ on the halfline to the left of the first eigenvalue of $S_{\Omega'}$. As regards the first method, we are concerned with the sesquilinear form D on $G(\Omega)$ defined by (5), for which one part of the Dirichlet principle is as follows:

Lemma 1. *Assume $\lambda < \lambda_1(S)$; let $u \in N(A)$ (the null space of A) be such that $u - g \in G_0(\Omega)$. Then*

$$D(u, u-g) = 0,$$

and

$$D(u) \leqslant D(g),$$

where equality holds if and only if $u = g$. The assertion remains valid if g is permitted to have discontinuous first derivatives on an $(m-1)$-dimensional submanifold of Ω.

Proof. The proof follows the line of the proof of the known Dirichlet principle, and may therefore be omitted.

In order to clarify that K is symmetrizable in the sense of Lax [6, p. 633], we require a space $\hat{X}^2(\Gamma)$ which we define to be the linear space $C(\Gamma)$ endowed with scalar product $(\,,\,)\hat{}$ given by

$$(f, g)\hat{} := (Jf, g) \qquad (f, g \in C(\Gamma)),$$

and the generated norm $\|\cdot\|\hat{}$ defined by

$$\|f\|\hat{} := ((f,f)\hat{})^{1/2} \qquad (f \in C(\Gamma)).$$

Further, let H be the completion of $\hat{X}^2(\Gamma)$. One can prove that in $\hat{X}^2(\Gamma)$ the operator K is symmetric (cf. [1]) and bounded (cf. [6, pp. 635, 636]), and thus it has a continuous extension $\tilde{K}$ onto H. We now prove

LEMMA 2. *Assume $\lambda < \lambda_1(S_{\Omega'})$. Then $\tilde{K}$ is symmetric and compact in H, and*

$$\sigma_P(K)\backslash\{0\} = \sigma_P(\tilde{K})\backslash\{0\}.$$

Proof. It follows from the symmetry of K in $\hat{X}^2(\Gamma)$ that $\tilde{K}$ is symmetric as well, since the scalar product in H is continuous. We have already mentioned that K is compact in $C(\Gamma)$ endowed with the maximum norm. By a result of Lax [6, Theorem 1] we can thus conclude that $\tilde{K}$ is compact. Furthermore, if $\sigma(K)$ denotes the spectrum of K as an operator in $C(\Gamma)$ with the maximum norm, we have

$$\sigma(K) = \sigma(\tilde{K}). \tag{11}$$

This is an application of [4, Theorem 5.20]. From (11) it follows in combination with the Theorem of F. Riesz that

$$\sigma_P(K)\backslash\{0\} = \sigma_p(\tilde{K})\backslash\{0\}. \tag{12}$$

The Lemma is therefore established.

Since K is symmetric in $\hat{X}^2(\Gamma)$, its eigenvalues are real, and we have

LEMMA 3. *Assume $\lambda < \lambda_1(S_{\Omega'})$. If $\mu \neq 0$ is an eigenvalue of K relative to Ω' with corresponding eigenfunction f, then*

$$\mu = (Kf, f)\hat{}/(f, f)\hat{} = [D_\Omega(Vf) - D_{X'}(Vf)]/[D_\Omega(Vf) + D_{X'}(Vf)].$$

This representation applies particularly in the potential–theoretic case, and offers an advantage over a known potential–theoretic formula derived by

Plemelj [8] in that it allows the localization of $\sigma_p(K)$ in the *open* interval $(-1, 1)$. We point out that the latter holds, in contrast to Plemelj's formula, even if the boundaries of the components of Ω are disconnected.

We can now prove

THEOREM 1. *Let Ω' and Ω'' be regular bounded open sets within an ε-neighbourhood of Ω and such that $\Omega' \Supset \Omega'' \Supset \Omega$. Assume $\lambda < \lambda_1(S_{\Omega'})$. If μ' and μ'' are dominant Fredholm eigenvalues for A_Ω relative to Ω' and Ω'' respectively, and if both are of the same sign, then*

$$\mu' > \mu''.$$

Remark. The above assertion holds on the understanding that the integral operators L' and L'', which are respectively associated with A_Ω relative to Ω' and Ω'', have been defined with the aid of the same positive function w on $\overline{\Omega'}$ satisfying $T_{\Omega'}(-(1/w)\nabla w) > 0$.

Proof of Theorem 1. We gather from the above discussion that the dominant Fredholm eigenvalues for A_Ω relative to Ω' and Ω'' are just the dominant eigenvalues of K' and K'' (the adjoints of L' and L'' in $X^2(\Gamma)$) respectively. These eigenvalues, μ' and μ'' say, are readily verified to be different from zero. Assume, without loss of generality, that they are positive, and that their corresponding eigenfunctions are f' and f''. It suffices to show that there is a lower bound ν of μ' satisfying $\nu > \mu''$, for then $\mu' \geqslant \nu > \mu''$. We demonstrate that there is a $g \in C(\Gamma)$ such that[1]

$$\nu := [D_\Omega(V'g) - D_{X'}(V'g)]/[D_\Omega(V'g) + D_{X'}(V'g)]$$

has the required properties. To this end, we take g as the solution to the equation

$$g + K'g = \partial^+ V''f'',$$

where ∂^+ denotes the modified exterior normal derivative on Γ composed with w appropriate to Ω'. Such a function g exists and is unique, because according to the remarks following Lemma 3 we have that $-1 \notin \sigma_p(K')$. Now put

$$u(y) := V'g(y) \qquad (y \in \Omega').$$

Then

$$\partial^+ u = \partial^+ V''f'',$$

[1] Here, and in the sequel, we abbreviate V relative to Ω' and Ω'' by V' and V'' respectively.

and this implies

$$u(y) = V''f''(y) \qquad (y \in \tilde{\Omega}),$$

which is easily seen with the aid of Eq. (4). Consequently, we have

$$D_\Omega(V'g) = D_\Omega(V''f''),$$

and by Lemma 1 we obtain

$$D_{X'}(V'g) = \int_{X'} \{|\rho_w V'g|^2 + h\,|V'g|^2\} < \int_{X''} \{|\rho_w V''f''|^2 + h\,|V''f''|^2\}$$
$$= D_{X''}(V''f''),$$

where X'' is the complement of $\tilde{\Omega}$ in Ω''. This shows that $\nu > \mu''$. We now prove that ν is a lower bound of μ'. Since μ' is a dominant eigenvalue of K', it follows by Lemma 2 that μ' is also a dominant eigenvalue of $\tilde{K}'$. Therefore, noting that $\tilde{K}'$ is a symmetric and compact operator, we can apply a Theorem of Hilbert (cf. Riesz and Sz.–Nagy [9, p. 217]) to conclude that the modulus of any dominant eigenvalue of $\tilde{K}'$ is equal to its norm. We obtain

$$\mu' = \max_{f \in H}[(\tilde{K}'f, f)\hat{}/(f, f)\hat{}] \geqslant (K'g, g)\hat{}/(g, g)\hat{} = \nu,$$

and the proof is therefore complete.

4. Application to the Solution of the Dirichlet Problem

Consider the Dirichlet problem for A_Ω which requires that we find a function $u \in N(A_\Omega)$ such that u is (on Γ) equal to a given continuous function. We propose to represent u in the form of a modified double-layer potential

$$Wf(y) := \int_\Gamma \partial\gamma'(\cdot, y) \cdot f \qquad (y \in \Omega \cup X'),$$

where f is a continuous surface distribution to be found. This leads to the problem of determining the resolvent $R(1, L)$ at the point $+1$ of the boundary integral operator L associated with A_Ω relative to Ω'. According to the remarks following Lemma 3, we know that $R(1, L)$ exists and can be expanded into a uniformly convergent Neumann–Liouville series. It is, however, desirable to derive an expansion of $R(1, L)$ into a bounded integral operator series, whose convergence is adjustable by altering Ω'. To this end, we put

$$\tilde{L} := \tfrac{1}{3}I + \tfrac{2}{3}L,$$

where I stands for the identity operator. We have $\sigma_p(\tilde{L}) \subset (-\frac{1}{3}, 1)$, and thus obtain the desired expansion as

$$R(1, L) = \tfrac{2}{3}R(1, \tilde{L}) = \tfrac{2}{3}\sum_{j=0}^{\infty} \tilde{L}^j.$$

In direct application of Theorem 1 we deduce

THEOREM 2. *Let Ω' and Ω'' be open sets satsifying the conditions of Theorem 1. Assume $\lambda < \lambda_1(S_{\Omega'})$. If r' and r'' are the spectral radii of $\tilde{L}'$ and $\tilde{L}''$, respectively, then $r' > r''$ if and only if $r' > \frac{1}{3}$.*

Let us point out that the special case $\lambda = 0$ is related to the work of Kress and Roach as was discussed above.

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